Faraday instability: Linear analysis for viscous fluids

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We present a linear stability analysis of parametrically excited surface waves for the case of viscous fluids. We show that the inclusion of viscosity leads to an extension of Mathieu's differential equation, which is valid for the case of inviscid fluids, in the form of an integrodifferential equation. We numerically solve this equation for the case of a single as well as a double frequency excitation.

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I. INTRODUCTION

Recently, the Faraday instability [1] of the surface of a vertically oscillating fluid layer has been investigated experimentally and interesting nonequilibrium patterns have been found [2–4]. Theoretical considerations have started by considering the case of inviscid fluids. Benjamin and Ursell [5] were the first to consider the linear stability analysis of the plane surface. They showed that the linear dynamics of the amplitudes of the surface modes is governed by Mathieu's equation. Viscous effects usually are included phenomenologically in the way discussed, e.g., by Landau and Lifschitz [6]. This treatment neglects the contributions that stem from rotational flow in the boundary layer close to the surface.

The aim of this paper is to present a treatment of the instability starting from the basic hydrodynamic equations including viscosity from the very beginning. Thereby, we consider the case of bulk viscosity since recent experiments [4] have been performed for highly viscous fluids. (For the discussion of effects of surface viscosity, we refer the reader to Eisenmenger [7].) We shall show that Mathieu's equation of the inviscid case is modified in two ways. First, the time derivatives ∂_t are substituted by the derivatives $\partial_t+2\nu k^2$, where ν denotes the viscosity of the fluid and k denotes the wave number of the surface wave. Second, a memory term arises leading to an integrodifferential equation. From this equation, it is possible to obtain the dispersion relation for the viscous surface waves of a nonoscillating fluid layer as given by Lamb [8].

The plan of this paper is as follows. First, we list the underlying basic equations of fluid dynamics. Then, we transform these equations to state variables that describe deviations from the state of the plane surface. Subsequently, the integrodifferential equation generalizing Mathieu's equation is given. Finally, we present numerical solutions.

II. THE BASIC EQUATIONS

In this section, we list the basic differential equations as well as the boundary conditions that govern the behavior of surface waves in the Faraday instability for the case of a viscous fluid.

The mathematical description of the surface waves has to take into account the motion of the fluid as well as the deformation of the free surface (cf., Fig. 1). The medium above the surface is air characterized by a constant atmospheric pressure $\tilde{p}$. The behavior of the fluid with viscosity ν and density ρ is described in a comoving coordinate system. We denote the coordinates in the horizontal directions by a vector $\bar{\mathbf{x}}:=(x,y)$. The x and y components of the velocity field $\mathbf{v}(\mathbf{x},z;t)$ are combined into the vector $\mathbf{v}^{(2)}=(v_x,v_y)$. Furthermore, we abbreviate the spatial derivatives by

$$\nabla=(\partial_x,\partial_y,\partial_z)\ ,\quad \nabla^{(2)}=(\partial_x,\partial_y)\ ,$$
$$\Delta=\partial_x^2+\partial_y^2+\partial_z^2\ ,\quad \Delta^{(2)}=\partial_x^2+\partial_y^2\ . \tag{1}$$

The fluid is assumed to be incompressible, i.e., the divergence of its velocity field vanishes: $\Delta\mathbf{v}(\mathbf{x},z,t)=0$. Since we take into account viscosity, the temporal evolution of the velocity field $\mathbf{v}(\mathbf{x},z,t)$ is governed by the Navier-Stokes equation (cf., Fig. 1);

$$\partial_t\mathbf{v}(\mathbf{x},z,t)+\mathbf{v}(\mathbf{x},z,t)\cdot\nabla\mathbf{v}(\mathbf{x},z,t)$$
$$=-\frac{1}{\rho}\nabla p(\mathbf{x},z,t)+\nu\Delta\mathbf{v}(\mathbf{x},z,t)-g[1-fW(\omega t)]\mathbf{e}_z\ . \tag{2}$$

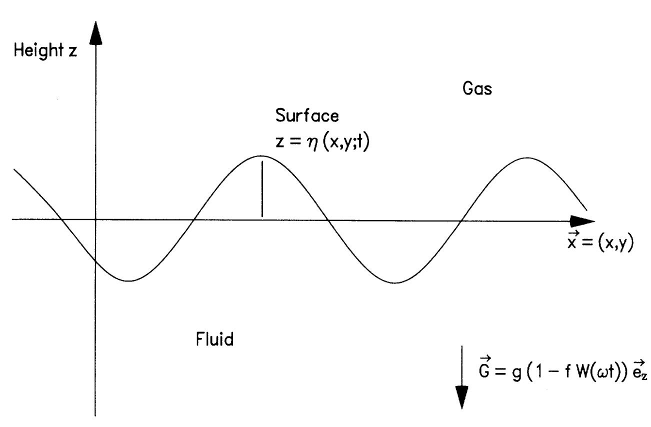

FIG. 1. Setup of the Faraday instability.

The time-dependent term $W(\omega t)$ stems from the temporal variation of the gravitational force $-g\mathbf{e}_z$ in the comoving coordinate system. This is caused by the vertical motion of the fluid layer with the excitation amplitude f. $p(\mathbf{x},z,t)$ describes the pressure in the fluid while the location of the surface is given by a function $z=\eta(\mathbf{x},t)$. Its dynamics is determined by the relationship which defines the vertical velocity component at the surface $z=\eta(\mathbf{x},t)$ as the substantial temporal derivative of the function $\eta(\mathbf{x},t)$, namely,

$$\frac{\partial\eta(\mathbf{x},t)}{\partial t}+\mathbf{v}^{(2)}(\mathbf{x},z=\eta(\mathbf{x},t),t)\cdot\nabla^{(2)}\eta(\mathbf{x},t)=v_z(\mathbf{x},z=\eta(\mathbf{x},t),t)\ . \quad (3)$$

In order to allow for a unique solution of the Navier-Stokes equation, one has to specify boundary conditions. Let us first consider the free surface $\eta(\mathbf{x},t)$ with normal vector $\mathbf{n}(\mathbf{x},t)$ (in outer direction), which is defined by

$$\mathbf{n}(\mathbf{x},t)=\frac{1}{\sqrt{1+(\nabla\eta)^2}}(-\partial_x\eta,-\partial_y\eta,1)\ . \quad (4)$$

Furthermore, let σ^{fl}denote the stress tensor of the fluid given by

$$\sigma_{ij}^{\mathrm{fl}}=-p\delta_{ij}+\rho\nu\left[\frac{\partial v_j}{\partial x_i}+\frac{\partial v_i}{\partial x_j}\right],\quad \delta_{ij}=\begin{cases}1, & \text{for } i=j\\ 0, & \text{else}\ ,\end{cases} \quad (5)$$

and σ^a the stress tensor of the air. σ^a is characterized by a constant atmospheric pressure $\tilde{p}$: $\sigma^a_{ij}=-\tilde{p}\delta_{ij}$. At the boundary $z=\eta(\mathbf{x},t)$, the difference between the stresses is balanced by surface tension [6],

$$\sum_{k=1}^{3}(-n_k\sigma^a_{ik}+n_k\sigma^{\mathrm{fl}}_{ik})=-\alpha\left[\frac{1}{R_1}+\frac{1}{R_2}\right]n_i\ ,\quad \text{for } i=1,2,3\ . \quad (6)$$

R_1 and R_2 are the curvature radii of the surface $z=\eta(\mathbf{x},t)$ given by

$$\left[\frac{1}{R_1}+\frac{1}{R_2}\right]=-\nabla\cdot\left[\frac{\nabla\eta(\mathbf{x},t)}{\sqrt{1+(\nabla\eta)^2}}\right]\approx-\Delta\eta(\mathbf{x},t)+\text{h.o.}, \quad (7)$$

where h.o. denotes higher-order terms.

Using the explicit expressions of the stress tensors, the boundary condition (6) reads

$$\left[\tilde{p}-p+\alpha\left[\frac{1}{R_1}+\frac{1}{R_2}\right]\right]n_i=-\nu\rho\sum_{j=1}^{3}\left[\frac{\partial v_j}{\partial x_i}+\frac{\partial v_i}{\partial x_j}\right]n_j\Bigg|_{z=\eta(\mathbf{x},t)},\quad i=1,2,3\ . \quad (8)$$

In addition to (8), boundary conditions have to be formulated at the bottom of the fluid vessel. One may use slip-free boundary conditions like $\mathbf{v}(\mathbf{x},z=-h,t)=\mathbf{0}$ or stress-free boundary conditions. For the sake of simplicity we used the latter ones. Therefore, we have to fulfill $v_z|_{z=-h}=0$ and $\partial_z^2 v_z|_{z=-h}=0$.

Finally, we mention that the consideration of a finite geometry needs analogous conditions to be stated at the lateral walls. However, we shall consider the case of periodic boundary conditions, which facilitate the mathematical treatment and should yield accurate results for the case of large aspect ratio systems.

III. BASIC STATE: PLANE SURFACE

In this section, we formulate the solution of the basic equations in the case of a plane surface. Then, we introduce dimensionless quantities and formulate the evolution equations for the variables describing the deviation from this basic state.

The state of a plane surface at $z=0$ of the fluid layer is given by $\eta(\mathbf{x},t)=0$, while the velocity field of the fluid is $\mathbf{v}(\mathbf{x},z,t)=0$. The integration of the Navier-Stokes equation becomes trivial and taking into account the boundary condition (8), one obtains

$$p_0(\mathbf{x},z,t)=\tilde{p}-\rho g[1-fW(\omega t)]z\ . \quad (9)$$

Now, we perform the transformation $p(\mathbf{x},z,t)=p_0(\mathbf{x},z,t)+\rho\pi(\mathbf{x},z,t)$ and introduce dimensionless quantities by the appropriate scaling of length and time. A natural time scale T is the period of the driving force $fW(\omega t)$: $T=1/\omega$ while we use $L=(\alpha/(\rho\omega^2))^{1/3}$ as the new length scale. Finally, we introduce the following dimensionless quantities:

$$\tilde{h}=\frac{1}{L}h,\quad \tilde{g}=\frac{T^2}{L}g,\quad \tilde{\nu}=\frac{T}{L^2}\nu,$$
$$\mathbf{v}=\frac{L}{T}\tilde{\mathbf{v}},\quad \eta=L\tilde{\eta},\quad \pi=\frac{L^2}{T^2}\tilde{\pi}\ , \quad (10)$$

ending up with the following set of equations. (For the sake of brevity we drop the tilde.) The evolution equation for the surface (3) remains unchanged and reads

$$\partial_t\eta(\mathbf{x},t)+\mathbf{v}^{(2)}(\mathbf{x},z=\eta(\mathbf{x},t),t)\cdot\nabla^{(2)}\eta(\mathbf{x},t)=v_3(\mathbf{x},z=\eta(\mathbf{x},t),t)\ . \quad (11)$$

The external force drops from the Navier-Stokes equation leading to

$$\partial_t\mathbf{v}(\mathbf{x},z,t)+\mathbf{v}(\mathbf{x},z,t)\cdot\nabla\mathbf{v}(\mathbf{x},z,t)=-\nabla\pi(\mathbf{x},z,t)+\nu\Delta\mathbf{v}(\mathbf{x},z,t)\ . \quad (12)$$

The boundary conditions (8), however, will depend on time explicitly due to the external forcing

$$\left[-\pi+g[1-fW(t)]\eta(\mathbf{x},t)+\left[\frac{1}{R_1}+\frac{1}{R_2}\right]\right]n_i=-\nu\sum_{k=1}^{3}\left[\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right]n_k\ ,\quad i=1,2,3,\quad z=\eta(\mathbf{x},t)\ . \quad (13)$$

We can summarize these boundary conditions as follows. Taking the scalar product of (13) with $\mathbf{n}$, we obtain

$$-\pi+g[1-fW(t)]\eta(\mathbf{x},t)+\left[\frac{1}{R_1}+\frac{1}{R_2}\right]=-2\nu\mathbf{n}\cdot(\mathbf{n}\cdot\nabla)\mathbf{v} . \quad (14)$$

Taking the vector product gives

$$\mathbf{n}\times[\mathbf{n}\times\nabla\times\mathbf{v}+2\mathbf{n}\cdot\nabla\mathbf{v}]=0 . \quad (15)$$

At the lower surface, boundary conditions for the velocity field have to be specified. The next step will be to analyze the stability of the plane surface, i.e., the basic state $\eta=0$, $\mathbf{v}=\mathbf{0}$.

IV. LINEARIZED EQUATIONS

The stability analysis of the basic state with plane surface requires a linearization of the evolution Eqs. (11) and (12) as well as the boundary conditions (13). The boundary conditions at the free surface $\eta(\mathbf{x},t)$ depend on the velocity field $\mathbf{v}(\mathbf{x},z;t)$ taken at $z=\eta(\mathbf{x},t)$. For small deformations of the surface, we may perform the following approximation:

$$\mathbf{v}(\mathbf{x},z=\eta(\mathbf{x},t),t)\approx\mathbf{v}(\mathbf{x},z=0,t)+\eta(\mathbf{x},t)\left[\frac{\partial\mathbf{v}(\mathbf{x},z,t)}{\partial z}\right]_{z=0}+\text{h.o.} \quad (16)$$

A similar expansion holds for the pressure term $\pi(\mathbf{x},z=\eta;t)$.

As usual we introduce the following decomposition of the velocity field:

$$\mathbf{v}(\mathbf{x},z,t)=\nabla\times\nabla\times\mathbf{e}_3S(\mathbf{x},z,t)+\nabla\times\mathbf{e}_3Z(\mathbf{x},z,t) . \quad (17)$$

The second term describes fluid motions that are purely horizontal. In the linear approximation, the corresponding modes do not couple to the surface deformations and decay due to viscosity. However, we emphasize that they may play an important role in the nonlinear regime. The evolution equation for the field $S(\mathbf{x},z,t)$ is obtained from the linearized Navier-Stokes equation (12). Taking twice the curl of this equation, one obtains

$$(\partial_t-\nu\Delta)\Delta\Delta^{(2)}S=0 . \quad (18)$$

The pressure can be calculated as follows:

$$\Delta^{(2)}\pi=-(\partial_t-\nu\Delta)\Delta^{(2)}\partial_zS . \quad (19)$$

The linearized evolution equation (3) for the surface reads

$$\partial_t\eta(\mathbf{x},t)+\Delta^{(2)}S(\mathbf{x},0,t)=0 . \quad (20)$$

The x and y components of the boundary conditions (13) take the form

$$(\partial_z^2-\Delta^{(2)})\nabla^{(2)}S(\mathbf{x},z,t)=\mathbf{0}|_{z=0} . \quad (21)$$

The z component leads to

$$\{-\pi(\mathbf{x},0,t)+g[1-fW(t)]\eta(\mathbf{x},t)-\Delta^{(2)}\eta(\mathbf{x},t)\}-2\nu[\partial_z\Delta^{(2)}S(\mathbf{x},z,t)]_{z=0}=0 . \quad (22)$$

The boundary conditions at the bottom are either (stress-free boundary)

$$S(\mathbf{x},z,t)=\partial_z^2S(\mathbf{x},z,t)=0|_{z=-h} , \quad (23)$$

or (slip-free boundary)

$$S(\mathbf{x},z,t)=\partial_zS(\mathbf{x},z,t)=0|_{z=-h} . \quad (24)$$

The standard treatment of the instability based on a neglection of viscosity (see [5]) is obtained from our equations by the limiting case $\nu=0$. Thereby, the boundary condition (21) drops out, together with the term $\nu\Delta$ in the Navier-Stokes equation. In that case, the velocity field can be assumed to be of potential type $(\Delta^{(2)}+\partial_z^2)S(\mathbf{x},z,t)=0$. However, for finite viscosity, this cannot hold close to the surface due to the boundary condition (21). As a result, a boundary layer close to the surface develops in the case of viscous fluids.

V. EXTENSION OF MATHIEU'S DIFFERENTIAL EQUATION BY THE INCLUSION OF DISSIPATION

The standard treatment of the Faraday instability neglects the viscosity of the fluid motion. In this case, Benjamin and Ursell [5] showed that the modes

$$\eta(\mathbf{x},t)=\eta(t)e^{i\mathbf{k}\cdot\mathbf{x}} \quad (25)$$

of the surface deformation obey Mathieu's equation

$$\partial_t^2\eta(t)+k\tanh(hk)[g+k^2+fgW(t)]\eta(t)=0 . \quad (26)$$

In this section we shall show how the correct inclusion of viscosity leads to a generalization of this equation. To this end, we start by making a plane wave ansatz for the field $S(\mathbf{x},z,t)$, the surface deformation $\eta(\mathbf{x},t)$, and the pressure π.

$$S(\mathbf{x},z,t)=[s_0(t)(e^{k\tilde{z}}-e^{-k\tilde{z}})+s_1(z,t)]e^{i\mathbf{k}\cdot\mathbf{x}} ,$$
$$\eta(\mathbf{x},t)=\eta(t)e^{i\mathbf{k}\cdot\mathbf{x}} , \quad (27)$$
$$\tilde{z}=z+h, \quad k=|\mathbf{k}| .$$

The first contribution to the field $S(\mathbf{x},t)$ is of potential type and, therefore, does not fulfill the boundary condition (21). However, it already fulfills the stress-free boundary conditions at the bottom of the vessel $(z=-h)$. The second contribution is assumed to obey the equation

$$[\partial_t-\nu(\partial_z^2-\mathbf{k}^2)]s_1(z,t)=0 , \quad (28)$$

so that the linearized Navier-Stokes equation is fulfilled. Furthermore, we require

$$[\partial_z^2+\mathbf{k}^2]s_1(z,t)|_{z=0}=-2\mathbf{k}^2(e^{kh}-e^{-kh})s_0(t) . \quad (29)$$

This condition ensures that $S(\mathbf{x},t)$ fulfills the boundary condition (21).

Inserting this ansatz into Eq. (22) and eliminating the pressure according to Eq. (19), we obtain

$$k(e^{kh}+e^{-kh})(\partial_t+2\nu\mathbf{k}^2)s_0(t)+2\nu\mathbf{k}^2\partial_z s_1(z,t)|_{z=0} +[g+\mathbf{k}^2+fgW(t)]\eta(t)=0 \;. \quad (30)$$

Now we can consider the linearized evolution equation for the surface,

$$\partial_t\eta(t)-\mathbf{k}^2[s_0(t)(e^{hk}-e^{-hk})+s_1(0,t)]=0 \;. \quad (31)$$

At the surface $z=0$, we can combine (28) and (29) to obtain

$$(\partial_t+2\nu\mathbf{k}^2)s_1(0,t)=-2\nu\mathbf{k}^2(e^{kh}-e^{-kh})s_0(t) \;. \quad (32)$$

This relation helps us to eliminate the variable $s_1(0,t)$ from the evolution Eq. (31),

$$\partial_t\{[\partial_t+2\nu\mathbf{k}^2]\eta(t)-\mathbf{k}^2(e^{kh}-e^{-kh})s_0(t)\}=0 \;. \quad (33)$$

We can integrate this relation once and obtain

$$\{[\partial_t+2\nu\mathbf{k}^2]\eta(t)-\mathbf{k}^2(e^{kh}-e^{-kh})s_0(t)\}=0 \;. \quad (34)$$

The resulting integration constant has to be put to zero since this would lead to a constant surface deformation without fluid motion.

Now we have to determine the quantity $\partial_z s_1(z;t)|_{z=0}$. To this end, we have to solve Eq. (28) together with the boundary condition (29). We introduce the transformation $s_1(z;t)=e^{-\nu k^2 t}\overline{s}_1(z;t)$ and $\eta(t)=e^{-\nu k^2 t}\overline{\eta}(t)$. Representing $\overline{s}_1(z,t)$ as a Fourier integral, we obtain the general solution for $\overline{s}_1(z;t)$;

$$\overline{s}_1(z,t)=\mathrm{Re}\int_0^\infty d\omega\,\overline{s}_1(\omega)\times(e^{\overline{k}(\omega)\overline{z}}-e^{-\overline{k}(\omega)\overline{z}})e^{i\omega t}\,,$$

$$\overline{k}(\omega)=\left[i\frac{\omega}{\nu}\right]^{1/2} \;. \quad (35)$$

This solution fulfills the stress-free boundary conditions $\overline{s}_1(z,t)=\partial_z^2\overline{s}_1(z,t)=0$ for $z=-h$. The amplitude $\overline{s}_1(\omega)$ is obtained from Eq. (28) together with (29), (32), and (34);

$$\overline{s}_1(\omega)=\frac{-2\nu}{e^{\overline{k}h}-e^{-\overline{k}h}}\overline{\eta}(\omega) \,, \quad (36)$$

where $\overline{\eta}(\omega)$ denotes the Fourier transform of $\overline{\eta}(t)$, leading to (c.f., Fig. 2)

$$\overline{s}_1(z,t)=-\int_{-\infty}^t dt'\,\mathrm{Re}\int_0^\infty d\omega\frac{1}{\pi}\frac{2\nu}{i\omega}\frac{(e^{\overline{k}\overline{z}}-e^{-\overline{k}\overline{z}})}{(e^{\overline{k}h}-e^{-\overline{k}h})}\times e^{i\omega(t-t')}\partial_t'\overline{\eta}(t') \;. \quad (37)$$

Now, it is straightforward to calculate $\partial_z s_1(z,t)|_{z=0}$;

$$\partial_z s_1(z,t)|_{z=0}=-\frac{2\sqrt{\nu}}{\pi}\int_{-\infty}^t dt'\,G(t-t')e^{-\nu k^2(t-t')}\times(\partial_{t'}+\nu k^2)\eta(t') \;. \quad (38)$$

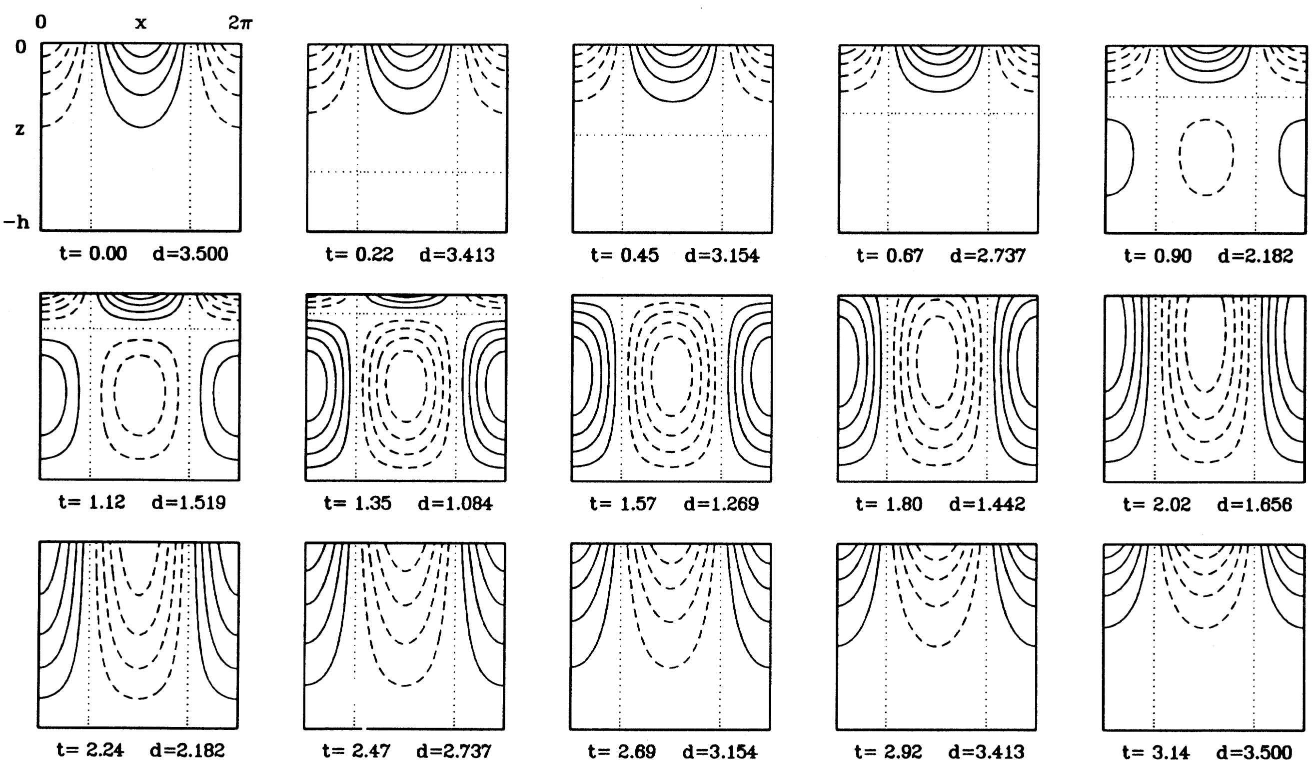

FIG. 2. Two-dimensional contour plot of the amplitude $s_1(x,z;t)$ over half a time period and one wavelength 2π. The dotted line represents $s_1=0$, the dashed line $s_1>0$, and the solid one $s_1<0$ in equidistant ($d/100$) contour lines.

The function $G(t-t')$ thereby is defined by the Fourier integral

$$G(t-t')=\mathrm{Re}\int_0^\infty d\omega \frac{1}{\sqrt{i\omega}\tanh[h\bar{k}(\omega)]}e^{i\omega(t-t')} . \quad (39)$$

It is impossible to calculate the integral (39) analytically. However, we can simplify the integral (39) for two limiting cases. First, we consider shallow water in the vessel,

$$\tanh(h\bar{k})\to h\bar{k} , \quad (40)$$

which leads to

$$G(t-t')=\frac{\pi\sqrt{\nu}}{2h} . \quad (41)$$

The second case is the deep water approximation where the height h of the fluid vessel is large with respect to the characteristic fluid length scale $1/k_c$. In this case, we can use

$$\tanh(h\bar{k})\to 1 . \quad (42)$$

We mention that in this case it becomes irrelevant whether one uses rigid or stress-free boundary conditions at the bottom of the vessel. Performing the Fourier integral (39), we end up with

$$G(t-t')=\sqrt{\pi/(t-t')} . \quad (43)$$

Due to Eq. (34), we can obtain a single equation for the amplitude $\eta(t)$ of the surface deformation;

$$\frac{1}{k\tanh(kh)}(\partial t+2\nu k^2)^2\eta(t)+[g+k^2+fgW(t)]\eta(t)-2\nu k^2\frac{2\sqrt{\nu}}{\pi}\int_{-\infty}^{t}dt'\,G(t-t')e^{-\nu k^2(t-t')}(\partial_{t'}+\nu k^2)\eta(t')=0 . \quad (44)$$

This is the starting point for an investigation of the linear Faraday instability including viscosity.

VI. NUMERICAL TREATMENT

The differential equation (44) can be treated only approximately. If we consider a time periodic forcing $W(t)$ with period π, the general solutions are superpositions of functions of the form

$$\eta(t)=e^{\lambda t}\tilde{\eta}(t) , \quad \tilde{\eta}(t)=\tilde{\eta}(t+2\pi) . \quad (45)$$

This is a result of Floquet's theory. For a numerical treatment, one expands the periodic function $\tilde{\eta}(t)$ into a Fourier series

$$\eta(t)=e^{\lambda t}\lim_{N\to\infty}\sum_{n=-N}^{N}\eta_n e^{int} . \quad (46)$$

Then, one obtains an infinite dimensional algebraic set of equations for the coefficients η_n;

$$\left[\frac{1}{k\tanh(kh)}(\lambda+in+2\nu k^2)^2-4\nu^2k^2\frac{\bar{k}_n}{\tanh(\bar{k}_n h)}+(g+k^2)\right]\eta_n-fg\sum_{n'=-N}^{N}W_{nn'}\eta_{n'}=0 . \quad (47)$$

The excitation term $W_{nn'}$, which stems from the time dependence of the driving force, is obtained from $W(t)$;

$$W_{nn'}=\frac{1}{2\pi}\int_0^T d\tau\, e^{i(n-n')\tau}W(\tau) , \quad (48)$$

where T is given by $W(t)=W(t+T)$.

A truncation of the ansatz (46) leads to a finite dimensional linear eigenvalue problem, which can be treated by numerical means.

VII. CLASSICAL PROBLEM: EXCITATION WITH $\cos(2\omega t)$

In this section we shall consider an excitation of the form $W(t)=\cos(2t)$ and use the lowest order truncation in Eq. (46). The ansatz (27) for the surface deformation now reads

$$\eta(t)=e^{\lambda t}(\eta e^{it}+\eta^* e^{-it}) . \quad (49)$$

Inserting this ansatz into (47), we obtain the following complex algebraic relation:

$$\eta\left[(g+k^2)+\frac{(\lambda+i+2\nu k^2)^2}{k\tanh(hk)}-2\nu k^2\frac{2\nu\sqrt{(\lambda+i/\nu+k^2)}}{\tanh(h\sqrt{(\lambda+i)/\nu+k^2})}\right]+\eta^*\frac{fg}{2}=0 . \quad (50)$$

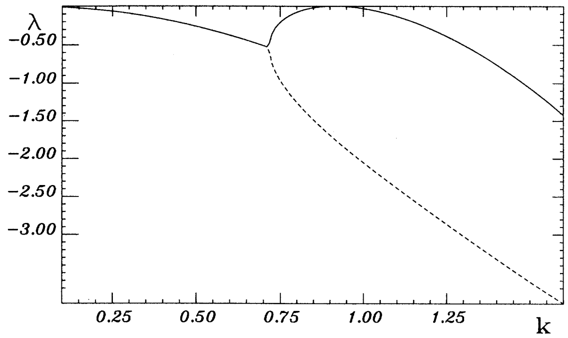

FIG. 3. Eigenvalues calculated from (51) for high viscosity in a single excitation mode.

The solvability condition for this equation leads to a relationship between the growth λ and the forcing amplitude f depending on the wave number k.

The onset of the Faraday instability is characterized by the condition $\lambda=0$. The solvability condition now reads

$$\frac{fg}{2}=\sqrt{|A|^2}\ , \tag{51}$$

while we have defined

$$A=(g+k^2)+\frac{1}{k\tanh(hk)}(2\nu k^2+i)^2 -2\nu k^2\frac{2\nu\sqrt{i/\nu+k^2}}{\tanh(h\sqrt{i/\nu+k^2})}\ . \tag{52}$$

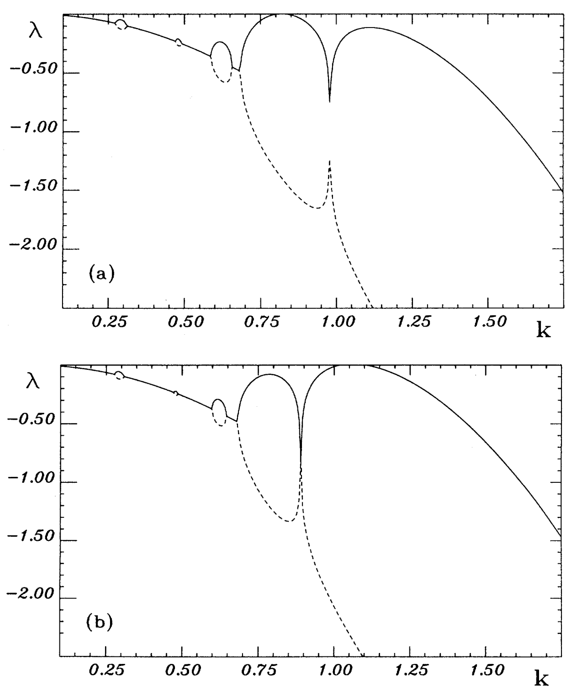

FIG. 4. Real part of the eigenvalue calculated from Eq. (47) with (54) for dual-frequency driving [4] with $\Phi=70°$, $\theta=70°$ (a) and $\Phi=70°$, $\theta=75°$ (b).

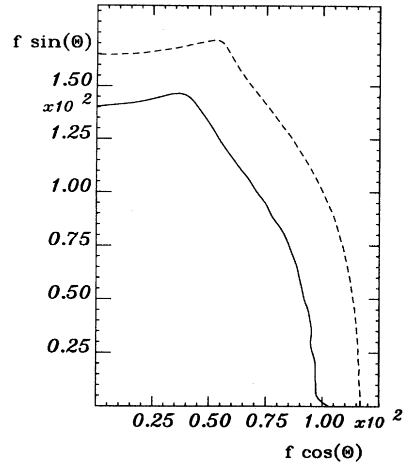

FIG. 5. Neutral curve for dual-frequency driving as measured by [4]: dashed line calculated for potential flow with phenomenological damping, e.g., [6], solid line obtained from Eq. (47) with (54) ($\Phi=70°$) in (m/s^2).

Since the quantity A depends on the wave number k, the relation (51) defines the neutral curve $f=f_c(k)$. The minimum of the neutral curve with respect to the wave number k yields the critical wave number k_c. The neutral curve essentially depends on the scaled viscosity ν and the scaled thickness h of the undisturbed fluid layer. Figure 3 shows the eigenvalues λ obtained from a numerical solution for a high value of the viscosity ν.

VIII. MULTIFREQUENCY EXCITATION

Recently, Edwards and Fauve performed an experiment [4] using an excitation with two commensurable frequencies of the form

$$W(t)=\cos(\theta)\cos(4t)+\sin(\theta)\cos(5t+\Phi)\ . \tag{53}$$

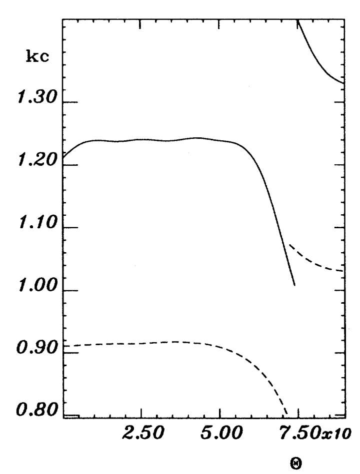

FIG. 6. Critical wave number $k_c(\theta,\Phi=70°)$: dashed line potential flow, solid line from Eq. (47), predicting the most unstable pattern wavelength.

They found spatially quasiperiodic patterns. The excitation leads to a matrix W of the form

$$W_{nn'}=\frac{1}{2}\cos(\theta)(\delta_{n,n'+4}+\delta_{n,n'-4})+\frac{1}{2}\sin(\theta)(e^{i\Phi}\delta_{n,n'+5}+e^{-i\Phi}\delta_{n,n'-5}) . \quad (54)$$

The solution of the corresponding eigenvalue problem yields eigenvalues λ, which depends on the forcing amplitude f, the angles θ and Φ, as well as on the wave number k. From this expression, we can calculate $f_c(k)$, e.g., the critical amplitude that denotes the instability of the plane surface with respect to disturbances with wave number k. The minimum of $f_c(k)$ with respect to k yields the critical wave number. This minimum is a function of θ and Φ and defines the neutral curves.

Figure 4 shows the real parts of the Floquet exponents as a function of k. As a result of the two frequencies of the excitation, there are basically two different wave numbers destabilizing, leading to bicriticality.

The neutral curve f_c depending on θ and Φ is exhibited in Fig. 5 in comparison with the neutral curve calculated for potential flow [5] presented as Edwards and Fauve [4] performed in their experiments. The corresponding values of the critical wave number is shown in Fig. 6 as a function of θ and Φ. It predicts a variety of patterns in the region where the critical wavelength is not determined by cos (4Φ) (left-hand side of Fig. 6), nor $\cos(5\Phi)$ (right-hand side). As a matter of fact Edwards and Fauve [4] found in this region of parameter-space quasiperiodic patterns.

IX. CONCLUSION

We have derived the basic hydrodynamical equations describing the parametric excitation of surface for the case of viscous fluids. We have shown that the linear stability analysis leads to a generalized Mathieu equation extending previous studies [5] for ideal fluids. The extended equations include a memory term that results from the existence of a viscous boundary layer close to the surface. We have performed analytical as well as numerical solutions of this equation.

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